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Comment on Bifurcations in Fluctuating Systems

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We prove, using normal form techniques in a codimension one bifurcation, that the conditional probability of the fast variable conditioned by the slow variables is a Gaussian distribution centered in the center manifold.

KEY WORDS: Bifurcation; stochastic differential equations; normal forms; center manifold.

In Ref. 1 an interesting method is proposed to eliminate the fast variables near a bifurcation point in a system of stochastic differential equations. The case of a codimension one instability (one eigenvalue zero of multiplicity one) is treated in detail for systems with two variables U_1 and U_2. The technique consists in working with the associated Fokker–Planck equation for the probability density $p_t(U_1, U_2)$ and postulating that this function admits the decomposition

$$p_t(U_1, U_2) = \left(\frac{\Gamma(U_1)}{\pi}\right)^{1/2} \exp\{-\Gamma(U_1)[U_2 - F(U_1)]^2\} \cdot \bar{p}_t(U_1) \qquad (1)$$

where U_1 is the critical variable corresponding to the eigenvalue zero, $U_2 = F(U_1)$ is the equation of the center manifold, and the width $\Gamma(U_1)$ of the Gaussian is determined by a self-consistency condition as a power series in U_1. The probability density $\bar{p}_t(U_1)$ obeys a reduced Fokker–Planck equation, which is calculated. However, it is necessary for the consistency of the method that the width $\Gamma(U_1)$ be a positive function of U_1 and it is this point that we shall discuss here. In Ref. 1, $\Gamma(U_1)$ was expanded up to

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linear terms in U_1 and the positivity was tacitly admitted. We shall prove, using the same normal form techniques of a previous note,[2] that the decomposition (1) holds with $\Gamma(U_1)>0$ provided that quadratic terms in the critical variable U_1 are retained. The method is a generalization to the stochastic case of some of the results obtained in Ref. 3 for the normal forms of singular vector fields.

We consider a vector $\mathbf{U}(t)=(U_1,..., U_N)$ that satisfies a system of stochastic differential equations (we use the notations of Ref. 2).

$$\partial_t \mathbf{U} = L\mathbf{U} + \sum_{r\geqslant 2} \mathbf{N}^{(r)}(\mathbf{U}) + \eta\left(\mathbf{D}(t) + L^{(1)}(t)\mathbf{U} + \sum_{r\geqslant 2} \mathbf{M}^{(r)}(t;\mathbf{U})\right) \tag{2}$$

where $\mathbf{U}=\sum_{\alpha=1}^{N} U_\alpha \mathbf{e}_\alpha$, $\mathbf{e}_1=(1,0,...,0)$, $\mathbf{e}_N=(0,...,0,1)$, L is a diagonal matrix $L\mathbf{e}_\alpha=\gamma_\alpha \mathbf{e}_\alpha$, $\gamma_1=0$, $\gamma_\alpha<0$, $\alpha\geqslant 2$, $L^{(1)}(t)$ is a matrix with elements $L^{(1)}(t)_{\alpha\beta}$, $\mathbf{D}(t)=\sum_{\alpha=1}^{N} D_\alpha(t)\,\mathbf{e}_\alpha$, and

$$\mathbf{N}^{(r)}(\mathbf{U}) = \sum_{\alpha,\alpha_j} u^{(r)}_{\alpha;\alpha_1,...,\alpha_r} U_{\alpha_1}\cdots U_{\alpha_r}\mathbf{e}_\alpha$$

$$\mathbf{M}^{(r)}(t;\mathbf{U}) = \sum_{\alpha,\alpha_j} v^{(r)}_{\alpha;\alpha_1,...,\alpha_r}(t)\, U_{\alpha_1}\cdots U_{\alpha_r}\mathbf{e}_\alpha$$

Here $D_\alpha(t)$, $L^{(1)}(t)_{\alpha\beta}$, and $v^{(r)}_{\alpha;\alpha_1,...,\alpha_r}(t)$ are Gaussian white noises with zero means and given correlations and η is the parameter measuring the intensity of the noise. In particular $\mathbf{D}(t)$ has correlations

$$\langle D_\alpha(t)\, D_\beta(t')\rangle = Q_{\alpha\beta}\delta(t-t')$$

The probability density $p_t(U_1, U_2,..., U_N)$ solution of the Fokker–Planck equation associated to (2) can be written as $p_t(U_2,..., U_N \mid U_1)$ $\bar{p}_t(U_1)$, where the first factor is the conditional probability of $(U_2,..., U_N)$ given U_1. We shall prove that this conditional probability can be taken for times $t\gg|\gamma_\alpha|^{-1}$, $\alpha\geqslant 2$, as a time-independent Gaussian distribution centered at the center manifold and with width depending on U_1, and also that $U_1(t)$ satisfies a closed stochastic differential equation, i.e., $\bar{p}_t(U_1)$ obeys a Fokker–Planck equation. The adiabatic elimination in (2) is done using the following ansatz (which we discuss at the end of this note): $\mathbf{U}$ is expressed asymptotically (for times $t\gg|\gamma_\alpha|^{-1}$) in terms of a critical variable C which obeys an autonomous equation. In formulas (up to first order in η)

$$\mathbf{U} = \sum_{r\geqslant 1} C^r \sum_{\alpha=1}^{N} U^{[r]}_\alpha \mathbf{e}_\alpha + \eta \sum_{r\geqslant 0} C^r \sum_{\alpha=1}^{N} V^{[r]}_\alpha(t)\,\mathbf{e}_\alpha \tag{3}$$

$$\partial_t C = \sum_{r\geqslant 1} f^{[r]} C^r + \eta \sum_{r\geqslant 0} g^{[r]}(t)\, C^r \tag{4}$$

The unknown constants $\{U_\alpha^{[r]}, f^{[r]}\}$ and the unknown stochastic processes $\{V_\alpha^{[r]}(t), g^{[r]}(t)\}$ in (3) and (4) are calculated by replacing (3) in the original equation (2) and using (4).[2,3] One obtains then, equating both sides of (2) at each order $[j, r]$, where $j=0$ or 1 is the order in η and r is the order in C, a sequence of homological equations for the unknowns. In order $[0, 1]$, Eq. (2) is satisfied with the choice $U_\alpha^{[1]}=\delta_{\alpha 1}$, $f^{[1]}=0$. In order $[0, r]$, one obtains an equation

$$-\gamma_\alpha U_\alpha^{[r]} = I_\alpha^{[r]} - f^{[r]}\delta_{\alpha_1} \tag{5}$$

where $I_\alpha^{[r]}$ depends only on $\{U_\alpha^{[s]}, f^{[s]}\}$ for $s<r$, which tells us that we can solve these equations by recursion in r to obtain

$$f^{[r]} = I_1^{[r]}, \qquad U_1^{[r]} = 0, \qquad U_\alpha^{[r]} = -\gamma_\alpha^{-1} I_\alpha^{[r]}, \qquad \alpha \geqslant 2$$

At the lowest order $[0, 2]$, one has

$$f^{[2]} = u_{1;11}^{(2)}, \qquad U_\alpha^{[2]} = -\gamma_\alpha^{-1} u_{\alpha;11}^{(2)} \equiv \rho_\alpha, \qquad \alpha \geqslant 2$$

In this way we have determined the unknown constants in the η-independent part of (3) and (4) and we have $U_1 = C$ (since $U_1^{[r]}=0$, $r \geqslant 2$), $U_\alpha = F_\alpha(C) + O(\eta)$, $\alpha \geqslant 2$, $\partial_t C = f(C) + O(\eta)$, where $\{F_\alpha(C), f(C)\}$ are now known as formal power series in C. We consider now the terms of order $[1, s]$, $s \geqslant 0$, and we obtain the equations

$$(\partial_t - \gamma_\alpha)\, V_\alpha^{[s]}(t) = J_\alpha^{[s]}(t) - g^{[s]}(t)\, \delta_{\alpha_1} \tag{6}$$

where $J_\alpha^{[s]}(t)$ depends on $\{U_\alpha^{[k]},\ k \leqslant s+1;\ f^{[k]},\ k \leqslant s\}$, which we know already, and on $\{V_\alpha^{[k]}(t), g^{[k]}(t); k<s\}$. Note that since we are calculating in the first order in η, the $J_\alpha^{[s]}(t)$ are linear in $V_\alpha^{[k]}(t)$ and $g^{[k]}(t)$. For instance, for $s=1$ one has

$$J_\alpha^{[1]}(t) = 2 \sum_{\beta=1}^{N} u_{\alpha;1\beta}^{(2)} V_\beta^{[0]}(t) + L^{(1)}(t)_{\alpha 1} - 2(1-\delta_{\alpha 1})\, \rho_\alpha\, g^{[0]}(t) \tag{7}$$

Since $\gamma_1 = 0$, we choose to solve (6) for $\alpha=1$, putting $g^{[s]}(t) = J_1^{[s]}(t)$ and $V_1^{[s]}(t) = 0$. Then

$$g^{[0]}(t) = D_1(t), \qquad g^{[1]}(t) = 2 \sum_{\alpha=1}^{N} u_{1;1\alpha}^{(2)} V_\alpha^{[0]}(t) + L^{(1)}(t), \qquad \text{etc.}$$

Next we consider Eqs. (6) for $2 \leqslant \alpha \leqslant N$ and for s up to some fixed r

$(0 \leqslant s \leqslant r)$. It is easy to see that these equations are of the form $(2 \leqslant \alpha \leqslant N$, $0 \leqslant s \leqslant r)$

$$(\partial_t - \gamma_\alpha) V_\alpha^{[s]}(t) - \sum_{k=0}^{s-1} \sum_{\beta=2}^{N} a_{\alpha\beta}^k V_\beta^{[k]}(t) = \xi_\alpha^s(t) \tag{8}$$

where $\xi_\alpha^s(t)$ is a white noise [a linear combination of the original white noises in (2)]. For $s=1$ one has

$$a_{\alpha\beta}^0 = -2u_{\alpha;1\beta}^{(2)}, \qquad \xi_\alpha^1 = -2\rho_\alpha D_1(t) + L^{(1)}(t)_{\alpha 1} \tag{9}$$

Up to $O(\eta C^r)$ Eqs. (2) are then replaced by (6) together with

$$U_1 = C \tag{10}$$

$$U_\alpha = F_\alpha(C) + \eta \sum_{s=0}^{r} C^s V_\alpha^{[s]}(t) \tag{11}$$

$$\partial_t C = f(C) + \eta \sum_{s=0}^{r} C^s g^{[s]}(t) \tag{12}$$

where in (12) each $g^{[s]}(t)$ depends linearly on the original white noises in (2) and on $\{V_\alpha^{[k]}(t), k < s\}$ solutions of (8).

From (10) and (11) we see that $U_\alpha = F_\alpha(U_1)$ is the equation of the center manifold. Putting $q_1 = V_2^{[0]}$, $q_2 = V_3^{[0]}, \ldots, q_N = V_2^{[1]}, \ldots, q_n = V_N^{[r]}$, where $n = (r+1)(N-1)$, we can write the system (8) in the form

$$\dot{q}_\mu(t) - \sum_{\nu=1}^{n} A_{\mu\nu} q_\nu = \xi_\mu(t), \qquad 1 \leqslant \mu \leqslant n \tag{13}$$

where the $n \times n$ matrix $A_{\mu\nu}$ has zero elements above the diagonal and $A_{11} = \gamma_2, \ldots, A_{NN} = \gamma_2, \ldots, A_{nn} = \gamma_N$, i.e.,

$$\det(A - \lambda) = \prod_{\alpha=2}^{N} (\gamma_\alpha - \lambda)^{r+1}$$

and all its eigenvalues are negative. In (13) the $\xi_\mu(t)$ are white noises with known correlations

$$\langle \xi_\mu(t)\, \xi_\nu(t') \rangle = R_{\mu\nu} \delta(t - t')$$

Then we solve (13) in the stationary state that exists $(\gamma_\alpha < 0)$ and the probability $p_t(q_1, \ldots, q_n)$ is time-independent and given by[4] the Gaussian distribution

$$p(\mathbf{q}) = \frac{1}{[(2\pi)^n \det \Xi]^{1/2}} \exp\left(-\frac{1}{2} \sum_{\mu,\nu=1}^{n} q_\mu \Xi_{\mu\nu}^{-1} q_\nu \right) \tag{14}$$

where the symmetric, positive-definite (generically) matrix Ξ satisfies $\Xi A^T + A\Xi + R = 0$ (A^T is the transposed matrix of A and R the matrix of elements $R_{\mu\nu}$). Using now (11), we can calculate the conditional probability $p(U_2,..., U_N | C)$, which is given by

$$p(U_2,..., U_N | C) = \int \prod_{s=0}^{r} \prod_{\alpha=2}^{N} dV_\alpha^{[s]}\, p(V_2^{[0]}, V_3^{[0]},..., V_N^{[0]}) \times \prod_{\alpha=2}^{N} \delta\left(\lambda_\alpha - \eta \sum_{s=0}^{r} C^s V_\alpha^{[s]}\right) \tag{15}$$

where $\lambda_\alpha \equiv U_\alpha - F_\alpha(C)$. We make now the change of variables

$$(q_1,..., q_n) \equiv (V_2^{[0]},..., V_N^{[r]}) \to (q'_1,..., q'_n) \equiv (s_2, s_3,..., s_N, V_2^{[1]}, V_3^{[1]},..., V_N^{[r]})$$

where

$$V_\alpha^{[0]} = s_\alpha - \sum_{s=1}^{r} C^s V_\alpha^{[s]}, \qquad 2 \leqslant \alpha \leqslant N$$

The Jacobian is one and putting $\bar{p}(\mathbf{q}') = p(\mathbf{q})$, we obtain from (15) after integration over $(s_2,..., s_N)$ the expression

$$p(U_2,..., U_N | C) = \eta^{-(N-1)} \int \prod_{s=1}^{r} \prod_{\alpha=2}^{N} dV_\alpha^{[s]} \times \bar{p}\left(\frac{\lambda_2}{\eta},..., \frac{\lambda_N}{\eta}, V_2^{[1]},..., V_N^{[r]}\right) \tag{16}$$

We note that $\bar{p}(\mathbf{q}')$ is still a Gaussian distribution ($\mathbf{q} \to \mathbf{q}'$ is a linear transformation) and then the last integral in (16) is just the marginal distribution of a Gaussian and consequently is a Gaussian in the variables $(\lambda_2/\eta,..., \lambda_N/\eta)$, which we write as

$$p(U_2,..., U_N | C) = \frac{1}{\eta^{N-1}[(2\pi)^{N-1} \det \Lambda(C)]^{1/2}} \times \exp\left[-\frac{1}{2\eta^2} \sum_{\alpha,\beta=2}^{N} \lambda_\alpha \Lambda(C)^{-1}_{\alpha\beta} \lambda_\beta\right] \tag{17}$$

where the positive-definite $(N-1) \times (N-1)$ matrix $\Lambda(C)$ is a function of C. The lowest order approximation corresponds to considering Eqs. (8) for $r = 0$, in which case they reduce to

$$(\partial_t - \gamma_\alpha)\, V_\alpha^{[0]}(t) = D_\alpha(t) \tag{18}$$

and one obtains for $\Lambda(C)$ the constant matrix $\Lambda_{\alpha\beta}=\delta_{\alpha\beta}Q_{\alpha\alpha}(2\,|\gamma_\alpha|)^{-1}$, a result also obtained in Refs. 2 and 5.

Using now (17), we have the following form for the probability density of the initial problem (2):

$$p_t(U_1,..., U_N)$$
$$=\frac{1}{[(2\pi\eta^2)^{N-1}\det\Lambda(U_1)]^{1/2}}$$
$$\times\exp\left\{-\frac{1}{2\eta^2}\sum_{\alpha,\beta=2}^{N}[U_\alpha-F_\alpha(U_1)]\,\Lambda(U_1)^{-1}_{\alpha\beta}[U_\beta-F_\beta(U_1)]\right\}\cdot\bar{p}_t(U_1) \tag{19}$$

with $\bar{p}_t(C)$ (we recall $U_1=C$) the probability density of the process defined by (12). We note that (18) is valid for times $t\gg\sup|\gamma_\alpha|^{-1}$, $\alpha\geqslant 2$, which is the condition of validity of our original ansatz (3) and (4). It should also be remarked that at this stage (12) is not a closed equation for C, since $g^{[s]}(t)$ depends on the original white noises in (2) and also on $\{V_\alpha^{[k]}(t), 2\leqslant\alpha\leqslant N, k<s\}$ [see after formula (7) for $g^{[0]}(t)$ and $g^{[1]}(t)$]. However, for times $t\gg\sup|\gamma_\alpha|^{-1}$ we can make a consistent white noise approximation as follows.

We first integrate (18) with initial conditions at $t=-\infty$, since we are in the stationary state [see (14)]

$$V_\alpha^{[0]}(t)=e^{\gamma_\alpha t}\int_{-\infty}^{t}dt'\,e^{-\gamma_\alpha t'}D_\alpha(t') \tag{20}$$

and we remark that for $t\gg|\gamma_\alpha|^{-1}$ we can replace $V_\alpha^{[0]}(t)\to|\gamma_\alpha|^{-1}D_\alpha(t)$ (see also Ref. 6). Then in Eqs. (8) for $s=1$ we use this replacement to obtain

$$(\partial_t-\gamma_\alpha)\,V_\alpha^{[1]}(t)=2\sum_{\beta=1}^{N}u^{(2)}_{\alpha;1\beta}\,|\gamma_\beta|^{-1}\,D_\beta(t)$$
$$+\,L^{(1)}(t)_{\alpha 1}-2\rho_\alpha D_1(t)\equiv D_\alpha^{(1)}(t) \tag{21}$$

where $D_\alpha^{(1)}(t)$ is a white noise. Integrating (21) as in (20), we can replace now $V_\alpha^{[1]}(t)\to|\gamma_\alpha|^{-1}D_\alpha^{(1)}(t)$ and we can proceed to write (8) for $s=2$ and so on. In this way we obtain from (12) an ordinary stochastic differential equation for C in which all the $g^{[s]}(t)$ are expressed in terms of the original white noises in (2).

In order to see how all this works, we give now the explicit calculations for $N=2$, $r=1$. Then Eqs. (11) reduce to $U_2=F_2(C)+$

$\eta(V_2^{[0]}(t)+CV_2^{[1]}(t))$ with $F_2(C)=\rho C^2+O(C^3)$, $\rho=|\gamma|^{-1}u_{2;11}^{(2)}$ (putting $\gamma=\gamma_2$), and Eqs. (13) become

$$\dot{q}_1-\gamma q_1=\xi_1(t), \qquad \dot{q}_2-\gamma q_2-aq_1=\xi_2(t) \tag{22}$$

where $a=2u_{2;12}^{(2)}$ and if we only have additive noise in (2), $\xi_1(t)=D_2(t)$, $\xi_2(t)=-2\rho D_1(t)$, which gives

$$R_{11}=Q_{22}, \qquad R_{12}=-2\rho Q_{12}, \qquad R_{22}=4\rho^2 Q_{11}$$

The matrix $\Xi_{\mu\nu}$ in (14) is of the form $(2|\gamma|)^{-1}\tilde{\Xi}$, with

$$\begin{aligned}\tilde{\Xi}_{11}&=R_{11}, \qquad \tilde{\Xi}_{12}=R_{12}+\frac{a}{2|\gamma|}R_{11}\\ \tilde{\Xi}_{22}&=R_{22}+\frac{a}{|\gamma|}R_{12}+\frac{a^2}{2\gamma^2}R_{11}\end{aligned} \tag{23}$$

and one easily checks that it is positive definite. Finally, (15) gives

$$p(U_2\,|\,C)=\frac{1}{[2\pi\eta^2\Lambda(C)]^{1/2}}\exp\left\{-\frac{[U_2-F_2(C)]^2}{2\eta^2\Lambda(C)}\right\} \tag{24}$$

with $\Lambda(C)=\Xi_{11}+2C\Xi_{12}+C^2\Xi_{22}>0$, since $\Xi_{\mu\nu}$ is positive definite. This case is explicitly treated in Ref. 1 using as an assumption that $p(U_2\,|\,C)$ is a Gaussian of the form in (1) [see formula (3.6) of Ref. 1], then developing $\Gamma(C)=\Gamma_0+\Gamma_1 C+O(C^2)$ and determining Γ_0 and Γ_1 by direct replacement in the Fokker–Planck equation associated to (2). One obtains then

$$\Gamma(C)=\frac{1}{2\eta^2\Xi_{11}}\left(1-2C\frac{\Xi_{12}}{\Xi_{11}}\right) \tag{25}$$

which indeed coincides with (24) when we develop there $\Lambda(C)$ up to order $O(C)$. However, in order to have $\Lambda(C)$ positive definite, the first correction to the leading order [which is $\Lambda(C)=\Xi_{11}$ in this case] must be quadratic and this follows naturally from the method exposed here. We remark that in the above reasoning no explicit statement on the order of magnitude of C is made. If we make the additional assumption that C is small in the vicinity of the bifurcation point and the concomitant restriction for the normalized probability distribution, then for the positivity of $\Lambda(C)$ it is sufficient to ensure that $\Lambda(C=0)$ is positive definite, which also follows from our analysis.

We discuss now briefly the ansatz (3) and (4). In order to study (2), we look for a nonlinear change of variables $(U_1,..., U_N) \to (C_1,..., C_N)$ of the form (up to first order in η)

$$\mathbf{U} = \sum_{\alpha=1}^{N} C_\alpha \mathbf{e}_\alpha + \sum_{r\geqslant 2} \mathbf{U}^{[r]} + \eta \sum_{r\geqslant 0} \mathbf{V}^{[r]}(t) \tag{26}$$

Here $\mathbf{U}^{[r]}$ [respectively $\mathbf{V}^{[r]}(t)$] is of order r in $(C_1,..., C_N)$ with constant coefficients to be determined (respectively, with coefficients that are stochastic processes to be determined). In the new variables Eqs. (2) take the form

$$\partial_t C_\alpha = \gamma_\alpha C_\alpha + \sum_{r\geqslant 2} F_\alpha^{[r]} + \eta \sum_{r\geqslant 0} G_\alpha^{[r]}(t) \tag{27}$$

where $F_\alpha^{[r]}$ [respectively, $G_\alpha^{[r]}(t)$] is of order r in $(C_1,..., C_N)$ with constant coefficients to be determined (respectively, with coefficients that are stochastic processes to be determined). To determine the unknown quantities, we proceed as before with the ansatz, i.e., we replace (26) in (2) and consider the equations obtained at each order $[j, r]$, where j is the order in η and r is now the order in $(C_1,..., C_N)$ (see Ref. 7, where the calculation is done in detail in the case of the Hopf bifurcation). In this way one obtains the set of homological equations (see Ref. 3 for the deterministic case)

$$(\Gamma - L)\,\mathbf{U}^{[r]} = \tilde{\mathbf{I}}^{[r]} - \sum_{\alpha=1}^{N} F_\alpha^{[r]} \mathbf{e}_\alpha, \qquad r \geqslant 2 \tag{28}$$

$$(\partial_t + \Gamma - L)\,\mathbf{V}^{[r]}(t) = \tilde{\mathbf{J}}^{[r]}(t) - \sum_{\alpha=1}^{N} G_\alpha^{[r]}(t)\,\mathbf{e}_\alpha, \qquad r \geqslant 0 \tag{29}$$

where ∂_t acts only on the time dependence of the coefficients in $\mathbf{V}^{[r]}(t)$ and

$$\Gamma \equiv \sum_{\alpha=2}^{N} \gamma_\alpha C_\alpha \frac{\partial}{\partial C_\alpha}$$

We determine now the unknown $\{F_\alpha^{[r]}, G_\alpha^{[r]}(t)\}$ in such a way as to be able to solve (28) and (29) for $\{\mathbf{U}^{[r]}, \mathbf{V}^{[r]}(t)\}$. In (28) we impose that the right-hand side belongs to $\mathrm{Ran}(\Gamma - L)$ (solvability condition) and in (29) we impose that the stochastic processes $V_\alpha^{[r]}(t)$ admit a stationary solution. One finds then that this gives the following normal form for the system (assuming nonresonant conditions between the eigenvalues γ_α [3,8]):

$$\partial_t C_1 = \sum_{r\geqslant 2} f^{[r]}(C_1)^r + \eta \sum_{r\geqslant 0} g^{[r]}(t)(C_1)^r \tag{30}$$

$$\partial_t C_\alpha = C_\alpha\left(\gamma_\alpha + \sum_{r\geqslant 2} f_\alpha^{[r]}(C_1)^r + \eta \sum_{r\geqslant 0} g_\alpha^{[r]}(t)(C_1)^r\right), \qquad \alpha \geqslant 2 \tag{31}$$

i.e., we recuperate Eq. (12) for $C_1 = C$ and we obtain the normal form of the equations for the C_α, $\alpha \geqslant 2$. Since in these equations we have C_α in factor (i.e., only multiplicative noise appears), $C_\alpha = 0$ is an invariant manifold, the center manifold, to which the variables C_α relax for $t \gg \sup |\gamma_\alpha|^{-1}$ [the stationary probability will be $p_{st}(C_\alpha) = \delta(C_\alpha)$] and we can put $C_\alpha = 0$, $\alpha \geqslant 2$, in (26) and we recuperate the ansatz (3) expressing **U** as a function of C. A detailed study of these methods for a general bifurcation of arbitrary codimension will be presented elsewhere.

We finally remark that in our calculations we have assumed that the original system (2) was to be interpreted in the Stratonovic sense and consequently we have used the normal rules of calculus. This is no restriction, since if the interpretation is the Ito or some intermediate one, we can always rewrite the equation as a Stratonovic system, using, for instance, the techniques in Ref. 9.

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